Free Response Practice #42
Calculator Permitted

Consider the trigonometric functions $f(\alpha) = \frac{\sec\alpha + \tan\alpha}{\sec\alpha - \tan\alpha}$ and $g(\alpha) = \frac{1 + 2\sin\alpha + \sin^2\alpha}{\cos^2\alpha}$.

a. Find the values of both $f\left(\frac{3\pi}{4}\right)$ and $g\left(\frac{3\pi}{4}\right)$. Round your values to the nearest thousandth and state what conclusions can be made about the domains and the graphs of $f(\alpha)$ and $g(\alpha)$.

$$f\left(\tfrac{3\pi}{4}\right) = \frac{\sec\left(\frac{3\pi}{4}\right) + \tan\left(\frac{3\pi}{4}\right)}{\sec\left(\frac{3\pi}{4}\right) - \tan\left(\frac{3\pi}{4}\right)}$$

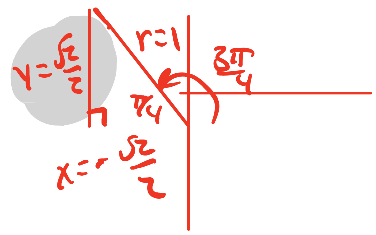
$$= \frac{-\sqrt{2} + (-1)}{-\sqrt{2} - (-1)}$$

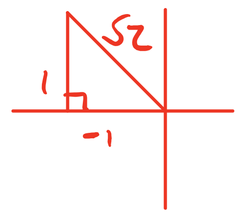
$$= \frac{-1(\sqrt{2}+1)}{-\sqrt{2}+1}$$

$$= \frac{-1(\sqrt{2}+1)}{-1(\sqrt{2}-1)}$$

$$f\left(\tfrac{3\pi}{4}\right) = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$f\left(\tfrac{3\pi}{4}\right) \approx 5.828$$





$$g\left(\tfrac{3\pi}{4}\right) = \frac{1 + 2\sin\left(\frac{3\pi}{4}\right) + \sin^2\left(\frac{3\pi}{4}\right)}{\cos^2\left(\frac{3\pi}{4}\right)}$$

$$= \frac{1 + 2\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)^2}{\left(-\frac{\sqrt{2}}{2}\right)^2}$$

$$= \frac{1 + \sqrt{2} + \frac{2}{4}}{\frac{2}{4}}$$

$$= \frac{1 + \sqrt{2} + \frac{1}{2}}{\frac{1}{2}} \quad \frac{(2)}{\cancel{(2)}}$$

$$= 2 + 2\sqrt{2} + 1$$

$$= 3 + 2\sqrt{2}$$

$$g\left(\tfrac{3\pi}{4}\right) \approx 5.828$$

Since $f\left(\frac{3\pi}{4}\right)$ and $g\left(\frac{3\pi}{4}\right)$ have numerical solutions, $\frac{3\pi}{4}$ is in the domain of g and f.

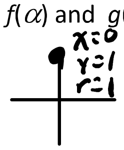
g and f must intersect at $\left(\frac{3\pi}{4}, 5.828\right)$

b. Find the values of $f\left(\frac{\pi}{2}\right)$ and $g\left(\frac{\pi}{2}\right)$. Round your values to the nearest thousandth and state what conclusions can be made about the domains and the graphs of $f(\alpha)$ and $g(\alpha)$.

$$f\left(\tfrac{\pi}{2}\right) = \frac{\sec\left(\frac{\pi}{2}\right) + \tan\left(\frac{\pi}{2}\right)}{\sec\left(\frac{\pi}{2}\right) - \tan\left(\frac{\pi}{2}\right)}$$

$$= \frac{\frac{1}{0} + \frac{1}{0}}{\frac{1}{0} - \frac{1}{0}}$$

$$f\left(\tfrac{\pi}{2}\right) = \text{und}$$



$$g\left(\tfrac{\pi}{2}\right) = \frac{1 + 2\sin\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{\pi}{2}\right)}{\cos^2\left(\frac{\pi}{2}\right)}$$

$$= \frac{1 + 2(1) + (1)^2}{0^2}$$

$$g\left(\tfrac{\pi}{2}\right) = \text{und}$$

$\frac{\pi}{2}$ is not in the domain of f or g.

The graphs of f and g are discontinuous at $\alpha = \pi/2$

c. Show, analytically, that the equation $\frac{\sec\alpha+\tan\alpha}{\sec\alpha-\tan\alpha}=\frac{1+2\sin\alpha+\sin^2\alpha}{\cos^2\alpha}$ is an identity for all values of α on the domain of each expression.

$$\frac{\frac{1}{\cos\alpha}+\frac{\sin\alpha}{\cos\alpha}}{\frac{1}{\cos\alpha}-\frac{\sin\alpha}{\cos\alpha}}=\frac{(1+\sin\alpha)^2}{\cos^2\alpha}$$

$$\frac{\cancel{\cos\alpha}}{\cancel{\cos\alpha}}\cdot\frac{\frac{1+\sin\alpha}{\cancel{\cos\alpha}}}{\frac{1-\sin\alpha}{\cancel{\cos\alpha}}}=\frac{(1+\sin\alpha)(1+\sin\alpha)}{1-\sin^2\alpha}$$

$$\frac{1+\sin\alpha}{1-\sin\alpha}=\frac{(1+\sin\alpha)\cancel{(1+\sin\alpha)}}{(1-\sin\alpha)\cancel{(1+\sin\alpha)}}$$

$$\frac{1+\sin\alpha}{1-\sin\alpha}=\frac{1+\sin\alpha}{1-\sin\alpha}$$

Free Response Practice #42 Grading Rubric

Free Response Part A – 2 points total

____ 1 Finds $f\left(\frac{3\pi}{4}\right) = g\left(\frac{3\pi}{4}\right) = 5.828$

____ 1 Since $f\left(\frac{3\pi}{4}\right)$ and $g\left(\frac{3\pi}{4}\right)$ are both defined, then $\frac{3\pi}{4}$ is in the domain of both functions and since $f\left(\frac{3\pi}{4}\right)$ and $g\left(\frac{3\pi}{4}\right)$ both equal the same value, then the graphs of $f(\alpha)$ and $g(\alpha)$ intersect each other at $\alpha = \frac{3\pi}{4}$.

Free Response Part B – 2 points total

____ 1 Finds $f\left(\frac{\pi}{2}\right) = g\left(\frac{\pi}{2}\right) =$ undefined

____ 1 Since $f\left(\frac{\pi}{2}\right)$ and $g\left(\frac{\pi}{2}\right)$ are both undefined, then $\frac{\pi}{2}$ is not in the domain of either function and since $f\left(\frac{\pi}{2}\right)$ and $g\left(\frac{\pi}{2}\right)$ are both undefined, then the graphs of $f(\alpha)$ and $g(\alpha)$ are discontinuous at $\alpha = \frac{\pi}{2}$.

Free Response Part D – 5 points total

____ 1 Rewrites the left side in terms of sine and cosine.

____ 1 Correctly simplifies the complex fraction on the left side to $\frac{1+\sin\alpha}{1-\sin\alpha}$

____ 1 Factors the numerator of $\frac{1+2\sin\alpha+\sin^2\alpha}{\cos^2\alpha}$ as (sinα + 1)(sinα + 1)

____ 1 Rewrites the denominator of $\frac{1+2\sin\alpha+\sin^2\alpha}{\cos^2\alpha}$ as $(1 - \sin^2\alpha)$ and then factors as $(1 + \sin\alpha)(1 - \sin\alpha)$

____ 1 Shows that each side simplifies to the same expression

$$\frac{\frac{1}{\cos\alpha}+\frac{\sin\alpha}{\cos\alpha}}{\frac{1}{\cos\alpha}-\frac{\sin\alpha}{\cos\alpha}} = \frac{(\sin\alpha+1)(\sin\alpha+1)}{1-\sin^2\alpha}$$

$$\frac{1+\sin\alpha}{\cos\alpha}\cdot\frac{\cos\alpha}{1-\sin\alpha} = \frac{(\sin\alpha+1)(\sin\alpha+1)}{(1+\sin\alpha)(1-\sin\alpha)}$$

$$\frac{1+\sin\alpha}{1-\sin\alpha} = \frac{\sin\alpha+1}{1-\sin\alpha}$$