Notes 10.3 Trigonometric Sum and Difference Identities

Sine Sum and Difference Identities

$$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

$$\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$$

Cosine Sum and Difference Identities

$$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$

$$\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$$

Tangent Sum and Difference Identities:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Consider for a moment the expression $\sin(45° + 60°)$. Obviously, this is equivalent to the expression $\sin 105°$, but the real question is does the expression also equal $\sin 45° + \sin 60°$? Use your calculator to find the value of both expressions.

$\sin 105° =$ 0.966

$\sin 45° + \sin 60° =$ 1.573

What conclusion can you reach based on the calculations above? Which one do you think is most likely the correct value and why? What explanation can you offer as to why the other expression does not give the correct value?

$\sin(45° + 60°) \neq \sin(45°) + \sin(60°)$

Use the formula sine sum identity to find the value of $\sin(45° + 60°)$. What do you notice?

$$\sin(45° + 60°) = \sin(45°)\cos(60°) + \sin(60°)\cos(45°)$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$
$$\sin(45° + 60°) = \frac{\sqrt{2}+\sqrt{6}}{4} \approx .960$$

1. Find the exact value of sin 15° using two angles from the unit circle whose difference is 15°.

$$\sin(15°) = \sin(60° - 45°)$$
$$= \sin(60°)\cos(45°) - \sin(45°)\cos(60°)$$
$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$
$$\sin(15°) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

2. Find the exact value of sin 285° using two angles from the unit circle whose sum is 285°.

$$\sin(285°) = \sin(240° + 45°)$$
$$= \sin(240°)\cos(45°) + \sin(45°)\cos(240°)$$

$$= -\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)$$
$$= \frac{-\sqrt{6}}{4} + \frac{-\sqrt{2}}{4}$$
$$= \frac{-\sqrt{6}-\sqrt{2}}{4}$$
$$\sin(285°) = -\frac{\sqrt{6}+\sqrt{2}}{4}$$

Simplify each of the following expressions using a sine sum or difference identity.

3. $sin\left(x+\frac{2\pi}{3}\right)$ = $\sin x \cdot \cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)\cos x$

$= \sin x \cdot \left(-\frac{1}{2}\right) + \frac{\sqrt{3}}{2}\cos x$

$= \frac{-\sin x}{2} + \frac{\sqrt{3}\cos x}{2}$

$= \frac{\sqrt{3}\cos x - \sin x}{2}$

4. $sin\left(\frac{\pi}{2}-\theta\right)$ = $\sin\left(\frac{\pi}{2}\right)\cdot\cos\theta - \sin\theta\cos\left(\frac{\pi}{2}\right)$

$= 1 \cdot \cos\theta - \sin\theta \cdot 0$

$= \cos\theta$

5. $\boxed{sin\,75° \, cos\,15° + cos\,75° \, sin\,15°}$ = $\sin(75° + 15°)$

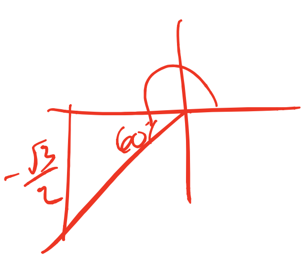
$= \sin(90°)$

$= 1$

6. $sin\,290° \, cos\,50° - cos\,290° \, sin\,50°$ = $\sin(290° - 50°)$

$= \sin(240°)$

$= -\sqrt{3}/2$



Cosine Sum and Difference Identities

$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

$\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

Use the sum or difference identity to find the exact value of the expression and then use the calculator to approximate the exact value expression.

7. $cos\left(\frac{\pi}{6}+\frac{\pi}{2}\right)$ = $\cos\frac{\pi}{6}\cos\frac{\pi}{2} - \sin\frac{\pi}{6}\sin\frac{\pi}{2}$

$= \frac{\sqrt{3}}{2}(0) - \left(\frac{1}{2}\right)(1)$

$= -\frac{1}{2}$

8. $cos(120° - 45°)$ = $\underbrace{\cos(120°)\cos(45°)} + \underbrace{\sin(120°)\sin(45°)}$

$= \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

Tangent Sum and Difference Identities

$$tan(A+B)=\frac{tan\,A+tan\,B}{1-tan\,A\,tan\,B}$$

$$tan(A-B)=\frac{tan\,A-tan\,B}{1+tan\,A\,tan\,B}$$

9. Find the exact value of tan 75° using two angles from the unit circle whose difference is 75°.

$$\tan(75°)=\tan(120°-45°)$$

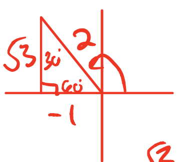
$$=\frac{\tan(120°)-\tan(45°)}{1+\tan(120°)\tan(45°)}$$

$$=\frac{-\sqrt{3}-1}{1+(-\sqrt{3})1}$$

$$=\frac{-\sqrt{3}-1}{1-\sqrt{3}}$$

$$=\frac{\cancel{-}(\sqrt{3}+1)}{\cancel{-}(-1+\sqrt{3})}$$

$$=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$



$\tan 120° = \frac{\sqrt{3}}{-1}$

10. Find the exact value of tan 165° using two angles from the unit circle whose sum is 165°.

$$\tan(165°)=\tan(120°+45°)$$

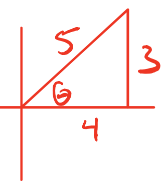
$$=\frac{\tan(120°)+\tan(45°)}{1-\tan(120°)\tan(45°)}$$

$$=\frac{-\sqrt{3}+1}{1-(-\sqrt{3})(1)}$$

$$=\frac{1-\sqrt{3}}{1+\sqrt{3}}$$

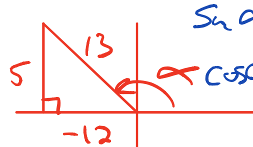
11. Suppose $sin\,\theta=\frac{3}{5}$ $\frac{=y}{=r}$ and $sin\,\alpha=\frac{5}{13}$ $\frac{=y}{=r}$.
Additionally, $tan\,\theta>0$ and $tan\,\alpha<0$.
Draw angles ϑ and α, labeling the reference triangles, and find the values of $sin(\theta+\alpha)$, $cos(\alpha-\theta)$, and $tan(\theta-\alpha)$. Show your work.

$\sin\theta=\frac{3}{5}$
$\cos\theta=\frac{4}{5}$



$\tan\theta=\frac{3}{4}$

$\sin\alpha=\frac{5}{13}$
$\cos\alpha=\frac{-12}{13}$



$\tan\alpha=\frac{5}{-12}$

$$\sin(\theta+\alpha)=\sin\theta\cos\alpha+\sin\alpha\cos\theta$$

$$=\frac{3}{5}\left(\frac{-12}{13}\right)+\left(\frac{5}{13}\right)\left(\frac{4}{5}\right)$$

$$=\frac{-36}{65}+\frac{20}{65}$$

$$\sin(\theta+\alpha)=\frac{-16}{65}$$

$$\cos(\alpha-\theta)=\cos\alpha\cos\theta+\sin\alpha\sin\theta$$

$$=\left(-\frac{12}{13}\right)\left(\frac{4}{5}\right)+\left(\frac{5}{13}\right)\left(\frac{3}{5}\right)$$

$$=\frac{-48}{65}+\frac{15}{65}$$

$$\cos(\alpha-\theta)=\frac{-33}{65}$$

$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$$

$\tan\theta=\frac{3}{4}$

$\tan\alpha=\frac{5}{-12}$

$$=\frac{\frac{3\cdot3}{3\cdot4}-\frac{5}{-12}}{1+\left(\frac{\cancel{3}^{1}}{4}\right)\left(\frac{5}{\cancel{-12}_{-4}}\right)}$$

$$=\frac{\frac{9}{12}+\frac{5}{12}}{\frac{16}{16}-\frac{5}{16}}$$

$$=\frac{\frac{14}{12}}{\frac{11}{16}}$$

$$=\frac{7}{\cancel{6}_{3}}\cdot\frac{\cancel{16}^{8}}{11}$$

$$\tan(\theta-\alpha)=\frac{56}{33}$$

Algebraically verify that the equations below are trigonometric identities.

12. $cos\left(\frac{\pi}{2} - x\right) = sin\, x$

$\cos\left(\frac{\pi}{2}\right)\cos x + \sin\left(\frac{\pi}{2}\right)\sin x =$

$0 \cdot \cos x + 1 \sin x =$

$\sin x = \sin x$

14. $tan\left(\frac{\pi}{2} - x\right) = cot\, x$

$\cancel{\dfrac{\tan\frac{\pi}{2} - \tan x}{1 + \tan\frac{\pi}{2}\tan x}} =$

$\cancel{\dfrac{\text{und} - \tan x}{1 + \text{und}\,\tan x}} =$

$\dfrac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} =$

$\dfrac{\sin\frac{\pi}{2}\cos x - \sin x \cdot \cos\frac{\pi}{2}}{\cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x} =$

$\dfrac{1 \cdot \cos x - \sin x\, 0}{0 \cdot \cos x + 1 \cdot \sin x} =$

$\dfrac{\cos x}{\sin x} =$

$\cot x = \cot x$

13. $tan\left(x + \frac{\pi}{4}\right) = \frac{cos\, x + sin\, x}{cos\, x - sin\, x}$

$\dfrac{\tan x + \tan\left(\frac{\pi}{4}\right)}{1 - \tan x \tan\left(\frac{\pi}{4}\right)} =$

$\dfrac{\tan x + 1}{1 - \tan x \cdot 1} =$

$\dfrac{\frac{\sin x}{\cos x} + \frac{\cos x}{\cos x}}{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}} =$

$\dfrac{\frac{\sin x + \cos x}{\cancel{\cos x}}}{\frac{\cos x - \sin x}{\cancel{\cos x}}} =$ $\cancel{\cos x}$ $\cancel{\cos x}$

$\dfrac{\sin x + \cos x}{\cos x - \sin x} =$

$\dfrac{\cos x + \sin x}{\cos x - \sin x} = \dfrac{\cos x + \sin x}{\cos x - \sin x}$

15. Simplify the expression below using the cosine sum and difference identities.

$cos\left(\frac{3\pi}{2} - x\right) + cos\left(\frac{3\pi}{2} + x\right)$

$= \cos\left(\frac{3\pi}{2}\right)\cos x + \sin\left(\frac{3\pi}{2}\right)\sin x + \cos\left(\frac{3\pi}{2}\right)\cos x - \sin\left(\frac{3\pi}{2}\right)\sin x$

$= 2\cos\left(\frac{3\pi}{2}\right)\cos x$

$= 2 \cdot 0 \cos x$

$= 0$