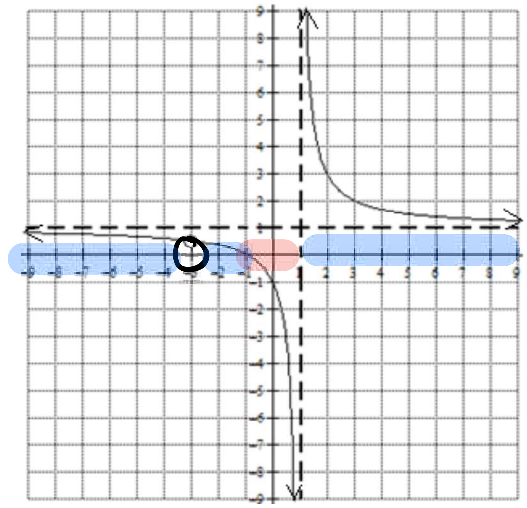
Notes 5.1 Solving Rational Equations and Inequalities

The graph of the rational function $g(x) = \frac{x^2+4x+3}{x^2+2x-3}$ is pictured to the right. Use the graph to identify the value(s) for which the following equations or inequalities are true?

1. $g(x) = 0$	$x = -1$
2. $g(x) > 0$	$(-\infty, -3) \cup (-3, -1) \cup (1, \infty)$
3. $g(x) < 0$	$(-1, 1)$
4. $g(x) \geq 0$	$(-\infty, -3) \cup (-3, -1] \cup (1, \infty)$
5. $g(x) \leq 0$	$[-1, 1)$



The goal of this lesson is to learn a strategy to algebraically solve rational inequalities. The process is identical to solving polynomial inequalities with one exception.

Given the function $g(x) = \frac{x^2+4x+3}{x^2+2x-3}$, find the function values $g(-3)$ and $g(1)$. Show your work.

$$g(-3) = \frac{(-3)^2+4(-3)+3}{(-3)^2+2(-3)-3} = \frac{9-12+3}{9-6-3} = \frac{0}{0} = \text{undefined}$$

$$g(1) = \frac{(1)^2+4(1)+3}{(1)^2+2(1)-3} = \frac{1+4+3}{1+2-3} = \frac{8}{0} = \text{undefined}$$

Using the graph above, explain why your efforts to find the values of $g(-3)$ and $g(1)$ had the outcome that they did.

At $x=-3$ and $x=1$, the graph of $g(x)$ doesn't have a defined y-vale.

Completely factor the function $g(x) = \frac{x^2+4x+3}{x^2+2x-3}$. Identify the value(s) of x which make the function equal zero and the value(s) of x which make the function undefined.

$$g(x) = \frac{\cancel{(x+3)}(x+1)}{\cancel{(x+3)}(x-1)}$$

we already know $x \neq -3, 1$

$g(x) = 0$: $\boxed{\begin{array}{c} x+1=0 \\ x=-1 \end{array}}$

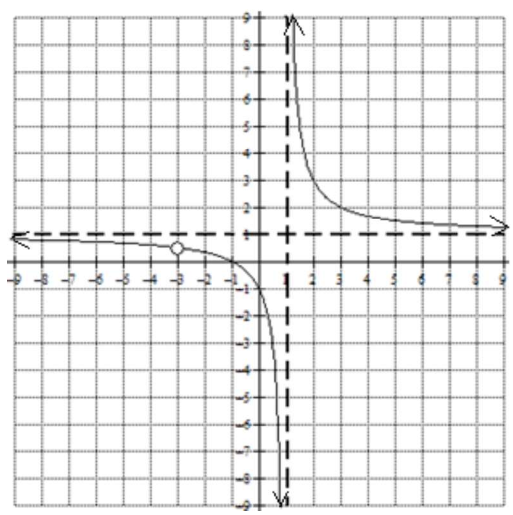
$g(x)$ is undefined: $\boxed{\begin{array}{c|c} x+3\neq 0 & x-1\neq 0 \\ x\neq -3 & x\neq 1 \end{array}}$

To solve rational inequalities, the process to follow is identical to the process previously learned to solve polynomial inequalities except in the case of rational inequalities, we must not only divide the number line for the sign analysis using values that make the function equal to zero but also using values that make the function undefined.

Now, consider the function $g(x) = \frac{x^2+4x+3}{x^2+2x-3}$.

1. Completely factor the given rational function, provided that it is not already factored.	$\frac{x^2+4x+3}{x^2+2x-3} \leq 0$ $\frac{(x+3)(x+1)}{(x+3)(x-1)} \leq 0$
2. Set each factor equal to zero to find the zeros of the function and the values where the function is undefined. Draw a number line that is divided into segments using the zeros.	$x \neq -3$ $x = -1$ $x \neq 1$ $x=-10$ $x=-2$ $x=0$ $x=10$ -3 -1 1
3. Choose a number from each interval and determine the sign of each factor of the rational based on the chosen value. This will enable you to determine if the entire rational is positive or negative on the interval in question.	$\frac{(-)(-)}{(-)(-)}$ $\frac{(+)(-)}{(+)(-)}$ $\frac{(+)(+)}{(+)(-)}$ $\frac{(+)(+)}{(+)(+)}$ Pos POS NEG Pos
4. Refer to the original inequality sign. In this particular example, the inequality symbol is $\leq$ which means the solution will be the values that make the polynomial negative or = 0. Write the solution interval(s).	≤ 0 means non-positive. which occurs on $[-1, 1)$

Now, refer to the graph of $g(x)$ pictured to the right. Explain how your sign analysis in the process above is confirmed by the graph.

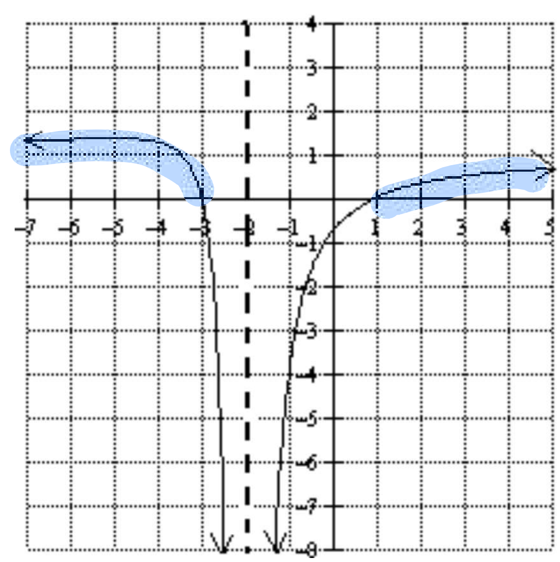


- $g(x)$ was positive and the graph of $g(x)$ is above the x-axis on the intervals $(-\infty, -3)$, $(-3, -1)$ and $(-1, \infty)$
- $g(x)$ was negative and the graph of $g(x)$ is below the x-axis on the interval $(-1, 1)$

1. Completely factor the given rational function, provided that it is not already factored.	$\frac{(x+3)(x-1)}{(x+2)^2} < 0$
2. Set each factor equal to zero to find the zeros of the function and the values where the function is undefined. Draw a number line that is divided into segments using the zeros.	$x = -3$ $x \neq -2$ $x = 1$ $x=-10$ | -3 | $x=-2.5$ | -2 | $x=0$ | 1 | $x=10$
3. Choose a number from each interval and determine the sign of each factor of the rational based on the chosen value. This will enable you to determine if the entire rational is positive or negative on the interval in question.	$\frac{(-)(-)}{+}$ POS | $\frac{(+)(-)}{+}$ NEG | $\frac{(+)(-)}{+}$ NEG | $\frac{(+)(+)}{+}$ POS
4. Refer to the original inequality sign. In this particular example, the inequality symbol is < which means the solution will be the values that make the polynomial negative. Write the solution interval(s).	< 0 means negative which occurs on $(-3,-2) \cup (-2,1)$

Now, refer back to the graph of $g(x)$ pictured below. Explain how your sign analysis in the process above is confirmed by the graph.

- The graph of $g(x)$ is above the x-axis and positive on $(-\infty,-3)$ and $(1,\infty)$
- The graph of $g(x)$ is below the x-axis and negative on $(-3,-2)$ and $(-2,1)$



Solve each of the following rational inequalities. Show your complete sign analysis and check your solution using a graphing calculator.

$$\frac{x-3}{x^2+4x+2} \geq 0$$

Disc $= b^2 - 4ac$
$= (4)^2 - 4(1)(2)$
$= 16 - 8$
Disc $= 8$

$x \neq \frac{-b \pm \sqrt{Disc}}{2a}$

$x \neq \frac{-4 \pm \sqrt{8}}{2(1)}$

$x \neq \frac{-4 \pm 2\sqrt{2}}{2}$

$x \neq -2 \pm \sqrt{2}$

$x = 3$

-3.4 -.6

$x=-10$ | $-2-\sqrt{2}$ | $x=-2$ | $-2+\sqrt{2}$ | $x=2$ | 3 | $x=4$

$\frac{(-)}{(+)}$ NEG | $\frac{(-)}{(-)}$ POS | $\frac{(-)}{(+)}$ NEG | $\frac{(+)}{(+)}$ POS

$\boxed{(-2-\sqrt{2}, -2+\sqrt{2}) \cup [3, \infty)}$

$$\frac{(2x+1)(x-3)^2}{(x+2)^2} < 0$$

$\frac{2(x+\frac{1}{2})(x-3)^2}{(x+2)^2} < 0$

$x \neq -2$ $x = -\frac{1}{2}$ $x = 3$

$x=-10$ | -2 | $x=-1$ | $-\frac{1}{2}$ | $x=0$ | 3 | $x=4$

$\frac{(-)+}{+}$ NEG | $\frac{(-)+}{+}$ NEG | $\frac{(+)+}{+}$ POS | $\frac{(+)+}{+}$ POS

$\boxed{(-\infty, -2) \cup (-2, -\frac{1}{2})}$

$$\frac{3}{x+2} \leq \frac{2}{x-1}$$

$\frac{(x-1)}{(x-1)} \frac{3}{x+2} - \frac{2}{x-1} \frac{(x+2)}{(x+2)} \leq 0$

$\frac{3x-3}{(x-1)(x+2)} - \frac{2x+4}{(x-1)(x+2)} \leq 0$

$\frac{x-7}{(x-1)(x+2)} \leq 0$

$x \neq -2$ $x \neq 1$ $x = 7$

$x=-10$ | -2 | $x=0$ | 1 | $x=5$ | 7 | $x=10$

$\frac{(-)}{(-)(-)}$ NEG | $\frac{(-)}{(-)(+)}$ POS | $\frac{(-)}{(+)(+)}$ NEG | $\frac{(+)}{(+)(+)}$ POS

$\boxed{(-\infty, -2) \cup (1, 7]}$