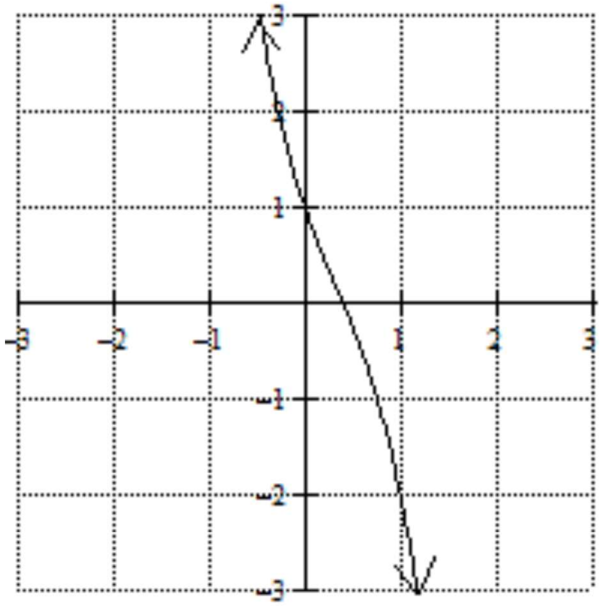
Notes 4.4 Analysis of Polynomial Functions
Numerical, Graphical and Analytical Approaches

The graph of a cubic function $g(x) = ax^3 + bx^2 + cx + d$ is pictured. Use the graph to answer the questions.



a. Based on the graph, what can be concluded about the values of a and d? Give reasons for your answers.

"a" is negative b/c the function rises on the left and falls on the right.

d = 1 b/c the y-int is the same as the constant, 1.

b. Based on the graph, what can be concluded about the roots of $g(x)$? Explain your reasoning.

By FTA, g(x) has 3 roots b/c it's cubic.

g(x) has one real root : positive with ODD multiplicity

b/c g(x) crosses the x-axis and may or may not change concavity.

c. Based on the graph, what is the maximum number of sign changes that the equation of $g(x)$ could possibly have? Give a reason for your answer.

g(x) only has one real zero and its positive so p(x) must have 1 or 3 sign chang.

d. If the coefficient of the linear term in the equation is -3 and $g(-1) = 8$, find the equation of $g(x)$.

$g(x) = ax^3 + bx^2 - 3x + 1$

$\boxed{g(-1) = 8}$

① $8 = a(-1)^3 + b(-1)^2 - 3(-1) + 1$

$8 = -a + b + 3 + 1$

$8 = -a + b + 4$

$4 = -a + b$

② From graph $\boxed{g(1) = -2}$

$-2 = a(1)^3 + b(1)^2 - 3(1) + 1$

$-2 = a + b - 3 + 1$

$-2 = a + b - 2$

$0 = a + b$

③ $4 = -a + b$

$0 = a + b$

$4 = 2b$

$2 = b$

④ $0 = a + b$

$0 = a + 2$

$-2 = a$

$$\boxed{g(x) = -2x^3 + 2x^2 - 3x + 1}$$

Consider the quartic $f(x) = -x^4 + 3x^2 - 2x - 2$ to answer the following questions.

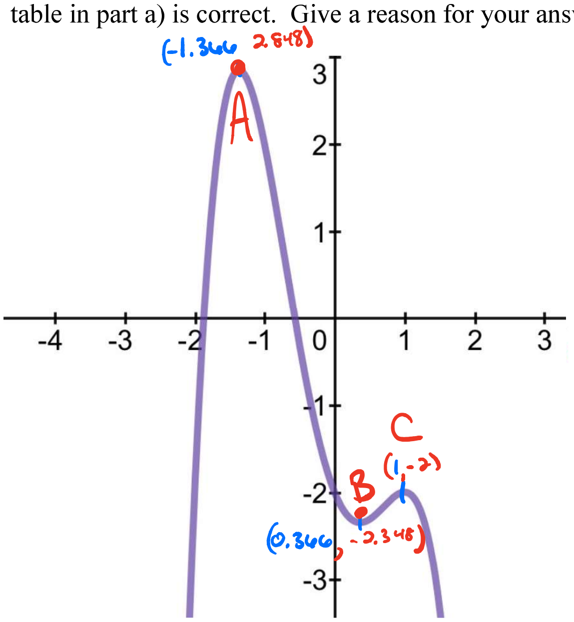
a. Use Descartes Rule of Signs to create a chart of possible combinations of the types of roots that $f(x)$ will have.

$$f(x) = -x^4 \underbrace{+ 3x^2 \underbrace{- 2x}_{1}}_{1} - 2 \quad 2 \text{ or } 0$$

$$f(-x) = \underbrace{-x^4 + 3x^2}_{1} + 2x \underbrace{- 2}_{1} \quad 2 \text{ or } 0$$

P	N	Zero	i
2	2	0	0
2	0	0	2
0	2	0	2
0	0	0	4

b. Sketch a graph of $f(x)$ on the axes pictured. Then, determine which combination of types of roots from your table in part a) is correct. Give a reason for your answer.



- f(x) is quartic so it has 4 zeros.
- Since the graph cross the negative x-axis twice w/o changing concavity, each negative root has multiplicity of 1.
- The remaining 2 roots must be imaginary.

c. Use the graphing calculator to determine the coordinates of the relative maximum(s) and relative minimum(s) of $f(x)$. Then, state the intervals on which $f(x)$ is increasing and decreasing.

Rel MAX: $(-1.366, 2.848)$ and $(1, -2)$

Rel MIN: $(0.366, -2.348)$

$f(x)$ is increasing on $(-\infty, -1.366) \cup (0.366, 1)$

$f(x)$ is decreasing on $(-1.366, 0.366) \cup (1, \infty)$

d. Find the coordinates of the points of inflection for the graph of $f(x)$. Then, state the interval(s) where the graph of $f(x)$ is concave up and concave down.

$$PoI \approx M_{\overline{AB}} \approx \left(\frac{\Sigma x}{2}, \frac{\Sigma y}{2}\right) \approx \left(\frac{-1.366 + 0.366}{2}, \frac{2.848 + {}^{-}2.348}{2}\right)$$

$$\approx \left(\frac{-1}{2}, \frac{0.5}{2}\right)$$

$$\approx \left(\frac{-1}{2}, \frac{1}{4}\right)$$

$$PoI \approx M_{\overline{BC}} = \left(\frac{\Sigma x}{2}, \frac{\Sigma y}{2}\right) \approx \left(\frac{1 + 0.366}{2}, \frac{-2.348 + (-2)}{2}\right)$$

$$\approx \left(\frac{1.0366}{2}, \frac{-4.348}{2}\right)$$

$$\approx (0.683, -2.174)$$

$f(x)$ is concave down on $(-\infty, -\frac{1}{2}) \cup (0.683, \infty)$

$f(x)$ is concave up on $(-\frac{1}{2}, 0.683)$

The equation of a cubic polynomial function is $f(x) = -2x^3 + 2x^2 + kx + 4$ and $f(2) = 0$.

a. Determine the left- and right-hand behavior of the graph of $f(x)$. Justify your answer.

- $f(x)$ is odd degree with NEGATIVE lead coefficient

$\therefore$ As $x \to -\infty$, $f(x) \to \infty$
As $x \to \infty$, $f(x) \to -\infty$

b. If $f(x)$ is divided by the factor $(x - 2)$, what will the remainder be? Justify your answer.

If $f(x) \div (x-2)$ the remainder will be 0 because $f(2) = 0$.

If a function is divided with a remainder of 0, then the divisor is a factor.

$\therefore$ $(x-2)$ is a factor of $f(x)$

c. Find the value of k in the equation of $f(x)$. Show your work.

$$\begin{array}{c|cccc} \boxed{2} & -2 & 2 & k & 4 \\ & 0 & -4 & -4 & 2k-8 \\ \hline & -2 & -2 & k-4 & \boxed{2k-4} \end{array}$$

$2k - 4 = 0$
$2k = 4$
$k = 2$

d. Determine all of the roots of $f(x)$. Show your work.

$$\begin{array}{c|cccc} \boxed{2} & -2 & 2 & 2 & 4 \\ & 0 & -4 & -4 & -4 \\ \hline & -2 & -2 & -2 & \boxed{0} \end{array}$$

$f(x) = (x-2)(-2x^2 - 2x - 2)$
$f(x) = (x-2)(-2)(x^2 + x + 1)$

$\text{Disc} = b^2 - 4ac$ $= (1)^2 - 4(1)(1)$ $= 1 - 4$ $\text{Disc} = -3$	$x = \dfrac{-b \pm \sqrt{\text{Disc}}}{2a}$ $x = \dfrac{-1 \pm \sqrt{-3}}{2(1)}$ $x = \dfrac{-1 \pm i\sqrt{3}}{2}$

$\boxed{\text{Roots} = 2,\ \dfrac{-1 - i\sqrt{3}}{2},\ \dfrac{-1 + i\sqrt{3}}{2}}$

Suppose that a polynomial function, $h(x)$, has roots of $x = -1$ with multiplicity of 1 and $x = 2i$.
What type of function is $h(x)$? Give a reason for your answer. By the Complex Conjugate Root Theorem, if $x=2i$ is a root, then $x=-2i$ is also a root. with $x=-1$ being a root, there are 3 roots. $\therefore$ By the FTA, $h(x)$ is cubic.
Find an equation, with integral coefficients, for $h(x)$. $h(x) = (x+1)(x-2i)(x+2i)$ $= (x+1)(x^2-2ix+2ix-4i^2)$ $= (x+1)(x^2-4(-1))$ $= (x+1)(x^2+4)$ $\boxed{h(x) = x^3+x^2+4x+4}$

Suppose that a polynomial function, $g(x)$, has roots of $x = 1$, which has a multiplicity of 2 and $x = 3 + i$.
What type of function is $g(x)$? Give a reason for your answer. • $x=1$ is a root of multiplicity 2. • By the Complex Conjugate Root Theorem, if $x=3+i$ is a root, then $x=3-i$ is a root. $\therefore g(x)$ has 4 roots $\therefore$ By FTA $g(x)$ is Quartic
Find an equation with, integral coefficients, for $g(x)$. $g(x) = (x-1)^2(x-(3+i))(x-(3-i))$ $=(x^2-2x+1)(x-3-i)(x-3+i)$ $=(x^2-2x+1)(x^2-3x-ix-3x+9+3i+ix-3i-i^2)$ $=(x^2-2x+1)(x^2-6x+9-(-1))$ $=(x^2-2x+1)(x^2-6x+10)$ $= x^4-2x^3+x^2-6x^3+12x^2-6x+10x^2-20x+10$ $\boxed{g(x) = x^4-8x^3+23x^2-26x+10}$