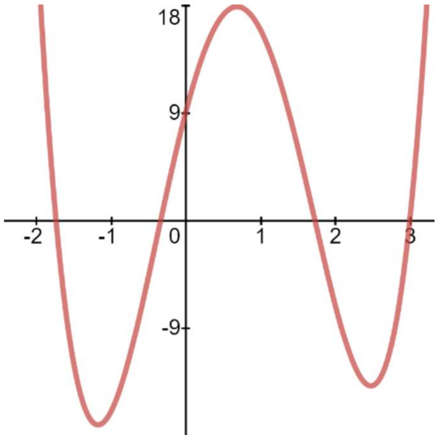
Notes 4.3 Using the Rational Root Theorem to Find Real and Imaginary Roots

Real roots can be one of two types:

RATIONAL or IRRATIONAL

Consider $h(x) = 3x^4 - 8x^3 - 12x^2 + 24x + 9$ whose graph is pictured below.



What three things about this graph make sense after investigating the equation of $h(x)$?

① The y-intercept is (0, 9) because the constant is 9.

② The graph rises on both sides b/c h(x) is even with lead coefficient positive

③ The degree of h(x) is 4 and the graph has 4 zeros each with multiplicity of 1

What conclusion can you draw about the four roots of the function, $h(x)$? Explain your reasoning.

Two roots are negative and two are positive, with each root having multiplicity 1 b/c h(x) crosses the x-axis w/o changing concavity at each root.

The goal of this lesson is to use the Rational Root Theorem to aid us in finding all of the roots of the function whether they be rational, irrational, or imaginary.

Make a list of the rational roots that the Rational Root Theorem guarantees are possible.

$$PRR = \frac{\pm 1, \pm 3, \pm 9}{\pm 1, \pm 3} =$$

$$= \pm 1, \pm 3, \pm 9, \pm \tfrac{1}{3}, \cancel{\pm \tfrac{3}{3}}, \cancel{\pm \tfrac{9}{3}}$$

From this list, which roots appear to be roots from the graph of $h(x)$? Perform synthetic division to verify that they are, in fact, roots. Divide $h(x)$ by one of the associated factors. Then, divide that result by the other associated factor.

$HPRR = 3, -\frac{1}{3}$

$\boxed{3}$	3	−8	−12	24	9
	0	9	3	−27	−9
$\boxed{-\frac{1}{3}}$	3	1	−9	−3	$\boxed{0}$
	0	−1	0	3	
	3	0	−9	$\boxed{0}$	

$$h(x) = (x-3)\left(x+\tfrac{1}{3}\right)(3x^2 - 9)$$

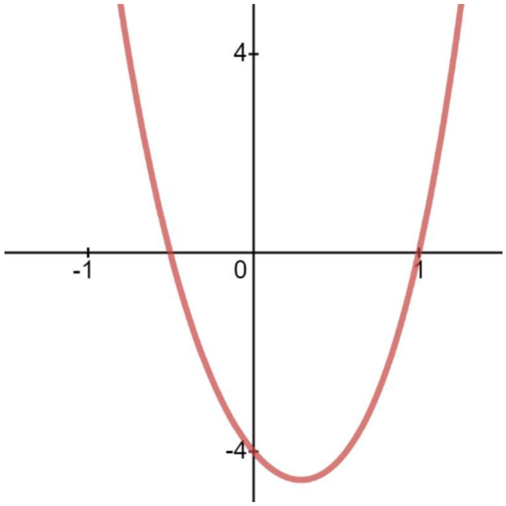
You have just divided a degree 4 function twice so your resulting polynomial is a QUADRATIC.

This function's roots will be the remaining two roots, which will be irrational, of $h(x)$. Find these irrational roots.

$$3x^2 - 9 = 0$$
$$3x^2 = 9$$
$$x^2 = 3$$
$$x = \pm\sqrt{3}$$

$$\boxed{\text{Roots: } -\sqrt{3}, -\tfrac{1}{3}, \sqrt{3}, 3}$$

Consider the function $f(x) = 2x^4 - x^3 + 7x^2 - 4x - 4$ whose graph is pictured below.



What do you notice about this graph that does **not** make sense based on the degree of the function?

The FTA guarantees 4 roots b/c the degree of f(x) is 4 but the graph has two zeros each with multiplicity of 1.

What conclusion can you draw about the four roots of the function, $f(x)$? Explain your reasoning.

- f(x) has two <u>real</u> roots : one positive
  one negative.
  each with multiplicity 1 b/c f(x) crosses the x-axis w/o changing concavity.
- The other two roots are imaginary.

Make a list of the rational roots that the Rational Root Theorem guarantees are possible.

$$PRR = \frac{\pm 1, \pm 2, \pm 4}{\pm 1, \pm 2}$$

$$PRR = \pm 1, \pm 2, \pm 4, \pm \tfrac{1}{2}$$

From this list, which roots appear to be roots from the graph above? Perform synthetic division to verify that they are, in fact, roots. Divide $f(x)$ by one of the associated factors. Then, divide that result by the other associated factor.

$HPRR = -\frac{1}{2}, 1$

$\boxed{1}$	2	-1	7	-4	-4
	0	2	1	8	4
$\boxed{-\frac{1}{2}}$	2	1	8	4	0
	0	-1	0	-4	
	2	0	8	0	

$$\boxed{f(x) = (x-1)(x+\tfrac{1}{2})(2x^2+8)}$$

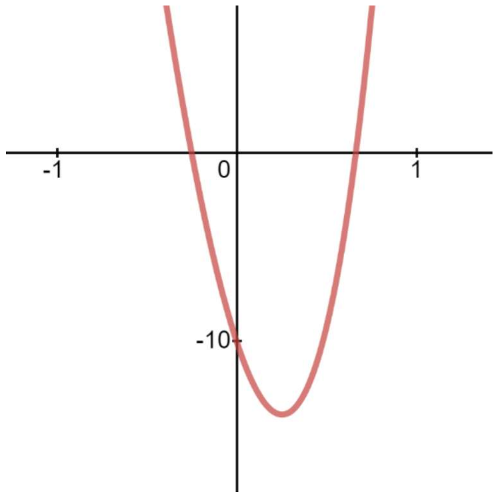
You have just divided a degree 4 function twice so your resulting polynomial is a QUADRATIC.

This function's roots will be the remaining two roots, which will be imaginary, of $f(x)$. Find these imaginary roots.

$$2x^2 + 8 = 0$$
$$2x^2 = -8$$
$$x^2 = -4$$
$$\sqrt{x^2} = \pm\sqrt{-4}$$
$$x = \pm 2i$$

$$\boxed{\text{Roots: } \pm 2i, -\tfrac{1}{2}, 1}$$

Consider the $f(x) = 12x^4 + 19x^3 + 48x^2 - 29x - 10$ graphed below to answer the following questions.



What conclusion can you make about the four roots of the function, $f(x)$? Explain your reasoning.

The FTA guarantees 4 roots b/c f(x) is 4th degree.

- f(x) has two real roots: one positive, one negative. each with multiplicity 1 b/c f(x) crosses the x-axis w/o changing concavity.
- The other two roots are imaginary.

Make a list of the rational roots that the Rational Root Theorem guarantees are possible.

$$PRR = \frac{\pm 1, \pm 2, \pm 5, \pm 10}{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}$$

$PRR = \pm 1, \pm 2, \pm 5, \pm 10, \pm \frac{1}{2}, \pm \frac{5}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{5}{3}, \pm \frac{10}{3}, \pm \frac{1}{4}, \pm \frac{5}{4}, \pm \frac{1}{6}, \pm \frac{5}{6}, \pm \frac{1}{12}, \pm \frac{5}{12}$

From this list, which roots appear to be roots from the graph above? Perform synthetic division to verify that they are, in fact, roots. Divide $f(x)$ by one of the associated factors. Then, divide that result by the other associated factor.

HPRR: $-\frac{1}{4}$ ✓, $\frac{2}{3}$

$\boxed{-\frac{1}{4}}$	12	19	48	-29	-10
	0	-3	-4	-11	10
$\boxed{2/3}$	12	16	44	-40	$\boxed{0}$
	0	8	16	40	
	12	24	60	$\boxed{0}$	

$f(x) = (x + \frac{1}{4})(x - \frac{2}{3})(12x^2 + 24x + 60)$

$= (x+\frac{1}{4})(x - \frac{2}{3})\,12(x^2 + 2x + 5)$

You have just divided a degree 4 function twice so your resulting polynomial is a QUADRATIC.

This function's roots will be the remaining two roots, which will be imaginary, of $f(x)$. Find these imaginary roots.

$x^2 + 2x + 5 = 0$

$DISC = b^2 - 4ac$
$= (2)^2 - 4(1)(5)$
$= 4 - 20$
$DISC = -16$

$x = \dfrac{-b \pm \sqrt{DISC}}{2a}$

$x = \dfrac{-(2) \pm \sqrt{-16}}{2(1)}$

$x = \dfrac{-2 \pm 4i}{2}$

$x = -1 \pm 2i$

$\boxed{\text{Roots}: -\frac{1}{4}, \frac{2}{3}, -1-2i, -1+2i}$

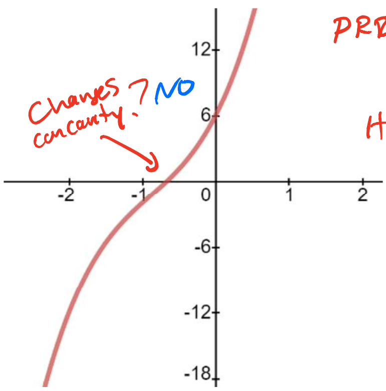
Based on the results of the previous pages, provide a plan on how to find all of the roots of a polynomial function.

Polynomial → FACTOR → Solve FACTOR = 0

Polynomial → PRR → Look at graph to make HPRR → Synthetically Divide Down to QUADRATIC → Solve QUAD

Find all roots, real and/or imaginary, of each of the following polynomial functions.

1. $g(x) = 3x^3 + 8x^2 + 13x + 6$



$PRR = \dfrac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 3}$

$HPRR = -2/3$

$\boxed{-2/3}$	3	8	13	6
	0	-2	-4	-6
	3	6	9	$\underline{\lvert 0}$

$g(x) = (x + 2/3)(3x^2 + 6x + 9)$

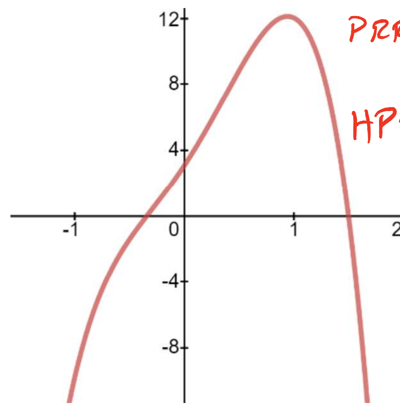
$g(x) = 3(x + 2/3)(x^2 + 2x + 3)$

$$\boxed{\begin{aligned} \text{Disc} &: b^2 - 4ac \\ &: (2)^2 - 4(1)(3) \\ &: 4 - 12 \\ \text{Disc} &: -8 \end{aligned}}$$

$$\boxed{\begin{aligned} x &= \frac{-b \pm \sqrt{\text{Disc}}}{2a} \\ &= \frac{-2 \pm \sqrt{-8}}{2(1)} \\ &= \frac{-2 \pm 2i\sqrt{2}}{2} \\ x &= -1 \pm i\sqrt{2} \end{aligned}}$$

$$\boxed{\text{Roots: } -2/3,\ -1 - i\sqrt{2},\ -1 + \sqrt{2}}$$

2. $h(x) = -6x^4 + x^3 + 4x^2 + 10x + 3$



$PRR = \dfrac{\pm 1, \pm 3}{\pm 1, \pm 2, \pm 3, \pm 6}$

$HPRR = \frac{3}{2}, -\frac{1}{3}$

$\boxed{3/2}$	-6	1	4	10	3
	0	-9	-12	-12	-3
$\boxed{-\frac{1}{3}}$	-6	-8	-8	-2	$\underline{\lvert 0}$
	0	2	2	2	
	-6	-6	-6	$\underline{\lvert 0}$	

$h(x) = (x - \frac{3}{2})(x + \frac{1}{3})(-6x^2 - 6x - 6)$

$h(x) = (x - \frac{3}{2})(x + \frac{1}{3})(-6)(x^2 + x + 1)$

$$\boxed{\begin{aligned} \text{Disc} &= b^2 - 4ac \\ &= (1)^2 - 4(1)(1) \\ &= 1 - 4 \\ \text{Disc} &= -3 \end{aligned}}$$

$$\boxed{\begin{aligned} x &= \frac{-b \pm \sqrt{\text{Disc}}}{2a} \\ x &= \frac{-(1) \pm \sqrt{-3}}{2(1)} \\ x &= \frac{-1 \pm i\sqrt{3}}{2} \end{aligned}}$$

$$\boxed{\text{Roots: } -\frac{1}{3},\ \frac{3}{2},\ \frac{-1 - i\sqrt{3}}{2},\ \frac{-1 + i\sqrt{3}}{2}}$$