Notes 4.1 Polynomial Functions that Have Imaginary Roots
Descartes' Rule of Signs

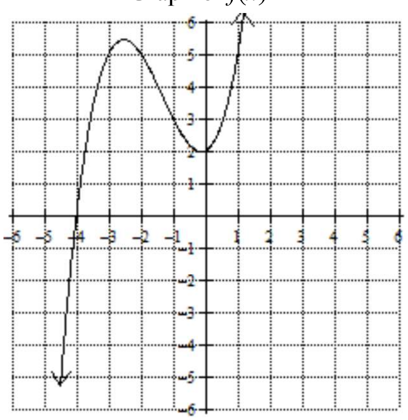
The Fundamental Theorem of Algebra

A polynomial function of "n" degree will have exactly "n" roots.

Roots of a polynomial function are positive, negative, zero, and/or imaginary

Consider the two functions that follow to answer the given questions. Answer each question addressing both functions, $f(x)$ and $g(x)$.

Graph of $f(x)$



If the graph is shifted up or down, what is the maximum number of zeros that the graph could have?

3

How many turning points does $f(x)$ have?

2

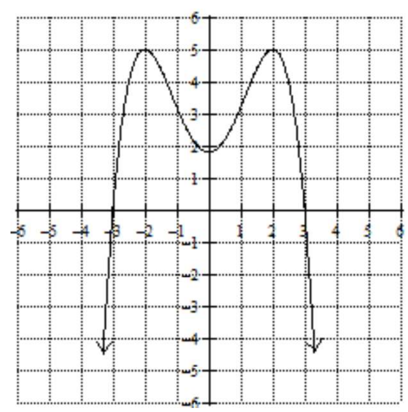
Based on the maximum number of zeros that the graph could have, what type of function is graphed?

likely Cubic

Based on the graph, identify the number of positive, negative, zero and imaginary roots of the function.

	$f(x)$
Positive	0
Negative	1
Zero	0
Imaginary	2

Graph of $g(x)$



If the graph is shifted up or down, what is the maximum number of zeros that the graph could have?

4

How many turning points does $g(x)$ have?

3

Based on the maximum number of zeros that the graph could have, what type of function is graphed?

Likely Quartic

Based on the graph, identify the number of positive, negative, zero and imaginary roots of the function.

	$g(x)$
Positive	1
Negative	1
Zero	0
Imaginary	2

How many of the roots can be POSITIVE and NEGATIVE?

Rene Descartes was a mathematician of his time to whom much of modern mathematics is accredited. One such concept that carries his name is Descartes' Rule of Signs.

Descartes' Rule of Signs

The maximum number of positive/negative roots of a polynomial function is equal to the number of sign changes of $f(x)/f(-x)$ or is less than that by an even amount.

If $f(x)$ had 7 sign changes, then the number of positive roots would be 7, 5, 3 or 1

If $f(-x)$ had 6 sign changes, then the number of negative roots would be 6, 4, 2 or 0

If $f(x)$ had 2 sign changes, then the number of positive roots would be 2 or 0

If $f(-x)$ had 1 sign changes, then the number of negative roots would be 1

Consider the function $p(x) = 2x^4 + x^3 - 8x^2 + 2x + 3$. Find the equation of $p(-x)$.

$p(-x) = 2x^4 - x^3 - 8x^2 - 2x + 3$

Descartes' Rule of Signs is dependent upon the number of times the signs of the terms in the equation change signs. In the boxes below, write both the equation of $p(x)$ and the equation of $p(-x)$. Then, count the number of times that the signs of the terms change.

$p(x) = 2x^4 + x^3 - 8x^2 + 2x + 3$

no yes yes no

The number of sign changes in $p(x)$ determines the MAXIMUM number of POSITIVE real roots that the function COULD have.

MAX 2

$p(-x) = 2x^4 - x^3 - 8x^2 - 2x + 3$

yes no no yes

The number of sign changes in $p(-x)$ determines the MAXIMUM number of NEGATIVE real roots that the function COULD have.

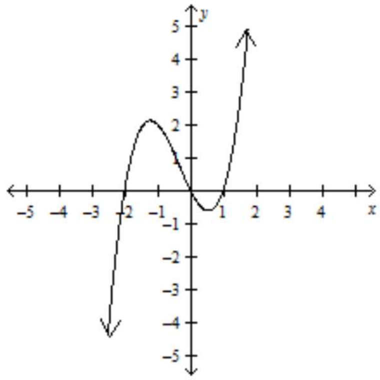
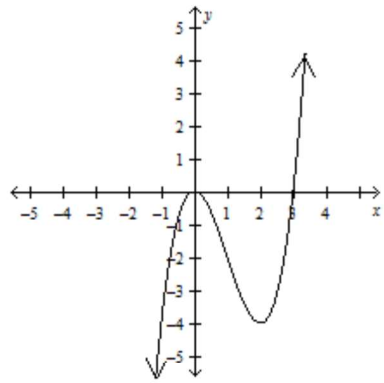
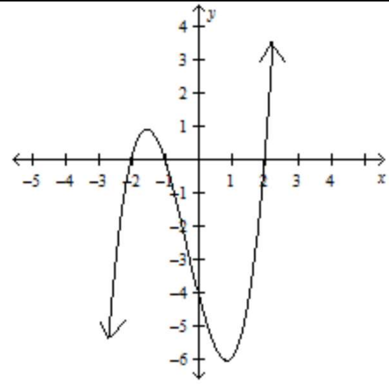
MAX 2

The function can have this maximum number of positive or negative roots or any multiple of two less this maximum. For example, if a function can have a maximum of 4 positive roots, then it could also possibly have 2 or 0 positive roots. If a function can have a maximum of 3 positive roots, then it could also have 1 positive root. If a function can have a maximum of 2 positive roots, then it could also have 0 positive roots.

For the function $p(x) = 2x^4 + x^3 - 8x^2 + 2x + 3$, how many POSITIVE roots are possible? 2 or 0

For the function $p(x) = 2x^4 + x^3 - 8x^2 + 2x + 3$, how many NEGATIVE roots are possible? 2 or 0

How many of the roots can be ZERO?

<table>
<tr><td></td><td></td><td></td></tr>
<tr><td>Graph of f(x)</td><td>Graph of g(x)</td><td>Graph of h(x)</td></tr>
<tr><td colspan="3">In the boxes below, rewrite each of the functions in completely factored form.</td></tr>
<tr><td>$f(x) = x^3 + x^2 - 2x$</td><td>$g(x) = x^3 - 3x^2$</td><td>$h(x) = x^3 + x^2 - 4x - 4$</td></tr>
<tr><td>$f(x) = x(x^2+x-2)$
$f(x) = x(x+2)(x-1)$</td><td>$g(x) = x^2(x-3)$</td><td>$h(x) = x^2(x+1) - 4(x+1)$
$h(x) = (x+1)(x^2-4)$
$h(x) = (x+1)(x-2)(x+2)$</td></tr>
<tr><td colspan="3">Based on the factors, how many times is x = 0 a root of the function? How is this visible in the graph?</td></tr>
<tr><td>x is a factor of f(x) one time.
∴ x=0 is a root once

f(x) crosses the x-axis without changing concavity.
∴ x=0 is odd mult = 1</td><td>x is a factor of g(x) twice
∴ x=0 is a root twice

g(x) is tangent to the x-axis at x=0
∴ x=0 is even mult ≥ 2</td><td>x is not a factor of h(x)
∴ x=0 is not a root of h(x)

h(x) does not contain the origin
∴ x=0 is not a root</td></tr>
</table>

Two of the three functions have something in common, both graphically and analytically, that is different from the other function. What are these differences?

- Graphically, f(x) and g(x) have x=0 as a root.
- Analytically, f(x) and g(x) are missing a constant term which allows a GCF of "x^n" to be factored out.

Based on what you've seen, how will you determine the number of times that $x = 0$ is a root of a function just by looking at the equation?

x=0 is a root "n" times where "x^n" is a GCF.

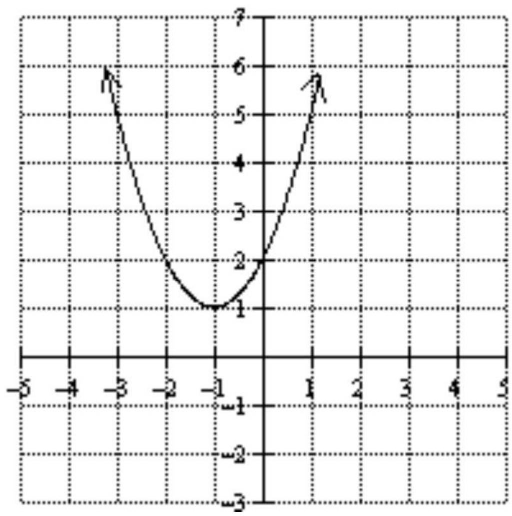
Is it possible that any of the roots of $p(x) = 2x^4 + x^3 - 8x^2 + 2x + 3$ are 0? Explain why or why not.

p(x) has a constant term which prevents a GCF of x^n. Thus, x=0 will not be a root of p(x)

$y = ax^2 + bx + c$

How many roots can be IMAGINARY?

Let's think back for a moment to Algebra II and quadratic functions. Consider the quadratic $f(x) = x^2 + 2x + 2$ whose graph is pictured. Notice that the graph does not show any zeros but since $f(x)$ is quadratic, then there MUST be two roots.



1. Find the value of $b^2 - 4ac$. What does this result tell you about the roots of $f(x)$.

Discriminant $= b^2 - 4ac$

$= (2)^2 - 4(1)(2)$

$= 4 - 8$

$= -4$

Since DISC < 0, $f(x)$ has imaginary roots.

2. Use the quadratic formula to find the roots of $f(x)$.

$x = \dfrac{-b \pm \sqrt{DISC}}{2a}$

$x = \dfrac{-2 \pm \sqrt{-4}}{2(1)}$

$x = \dfrac{-2 \pm 2i}{2}$

$x = -1 \pm i$

$x = -1 - i, -1 + i$

Since imaginary roots come from taking square roots of negatives in using the quadratic formula, what can be said about the possible number of imaginary roots of a polynomial function?

Imaginary roots must occur in even pairs of conjugates.

Complex Conjugate Root Theorem

If $P(x)$ is a polynomial with real coefficients, and $a + bi$ is a root of $P(x)$, then $a - bi$ is also a root of $P(x)$.

For the function $p(x) = 2x^4 + x^3 - 8x^2 + 2x + 3$, what are the possible numbers of imaginary roots?

Since $p(x)$ is quartic, there are 4, 2 or 0 imaginary roots.

Now, you have gathered all the information needed in order to determine the possible combinations of different types of roots. Draw a table below that summarizes the possibilities of the types of roots for $p(x)$. Then, graph the function on the calculator and put a star by the combination that is correct.

$P(x) = 2x^4 + x^3 - 8x^2 + 2x + 3$

Possible number of positive roots: 2 or 0

Possible number of negative roots: 2 or 0

Number of zero roots: 0

Possible number of imaginary roots: 4, 2 or 0

$p(x) = 2x^4 + x^3 - 8x^2 + 2x + 3$

$p(-x) = 2x^4 - x^3 - 8x^2 - 2x + 3$

P	N	Zero	i
2	2	0	0
2	0	0	2
0	2	0	2
0	0	0	4

For each of the functions below, use Descartes' Rule of Signs to help determine all of the possible combinations of positive, negative, zero, and imaginary roots that the function can have. Then, graph the function on a graphing calculator and put a star next to the combination of roots that is correct based on the graph.

1. $f(x) = -2x^5 + 2x^4 - 3x^3 + 2x^2$

1 1 1

$p(-x) = 2x^5 + 2x^4 + 3x^3 + 2x^2$

$+ = 3, 1$
$- = 0$
$\text{zero} = 2$
$i = 4, 2, 0$

P	N	Zero	i
3	0	2	0
☆ 1	0	2	2

2. $p(x) = -2x^4 + x^3 - 7x^2 + 4x + 1$

1 1 1

$p(-x) = -2x^4 - x^3 - 7x^2 - 4x + 1$

1

$+ = 3 \text{ or } 1$
$- = 1$
$0 = 0$
$i = 4, 2 \text{ or } 0$

P	N	Zero	i
3	1	0	0
☆ 1	1	0	2

3. $q(x) = 2x^3 - 3x^2 + 2x + 1$

1 2

$q(-x) = -2x^3 - 3x^2 - 2x + 1$

1

$+ = 2 \text{ or } 0$
$- = 1$
$0 = 0$
$i = 2 \text{ or } 0$

P	N	Zero	i
2	1	0	0
0	1	0	2

4. $g(x) = 3x^4 - 2x^3 - x^2 + 2x - 5$

1 1 1

$g(-x) = 3x^4 + 2x^3 - x^2 - 2x - 5$

1

$+ = 3 \text{ or } 1$
$- = 1$
$0 = 0$
$i = 4, 2 \text{ or } 0$

P	N	Zero	i
3	1	0	0
☆ 1	1	0	2
	1		
	1		