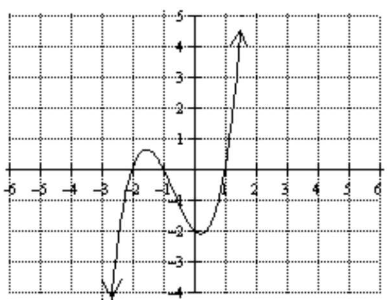
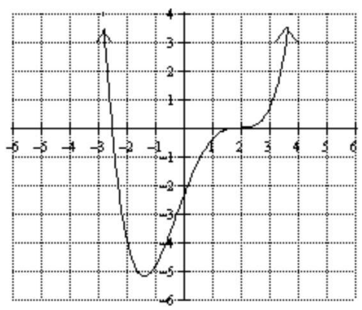
Notes 3.4 Zeros of Polynomial Functions
Analytical and Graphical Approaches

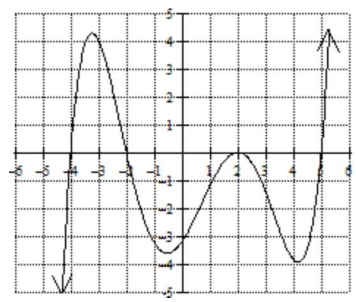
For each of the functions graphed below, state the end behavior and the zeros of the functions.



Graph of $f(x)$



Graph of $g(x)$



Graph of $h(x)$

$f(x)$ is a cubic function.	**$g(x)$ is a quartic function.**	**$h(x)$ is a quintic function.**
Left End Behavior $\lim_{x\to-\infty} f(x) = -\infty$	Left End Behavior $\lim_{x\to-\infty} f(x) = \infty$	Left End Behavior $\lim_{x\to-\infty} f(x) = -\infty$
Right End Behavior $\lim_{x\to\infty} f(x) = \infty$	Right End Behavior $\lim_{x\to\infty} f(x) = \infty$	Right End Behavior $\lim_{x\to\infty} f(x) = \infty$
Zeros $x = -2, -1, 1$	Zeros $x \approx -2.5, 2$	Zeros $x = -4, -2, 2, 5$

Define the **multiplicity** of a zero.

The multiplicity of a zero, $x=a$, is the # of times $(x-a)$ is a factor of the function

Read the following information about the multiplicities of the zeros of $f(x)$, $g(x)$, and $h(x)$ while studying the graphs above. Then, answer the questions on the next page.

On $f(x)$, all of the zeros have a multiplicity of 1.

On $g(x)$, the zero of $x = -2.5$ has a multiplicity of 1 and $x = 2$ has a multiplicity of 3.

On $h(x)$, the zeros $x = -4$, $x = -2$, and $x = 5$ have a multiplicity of 1 and $x = 2$ has a multiplicity of 2.

1. What do you notice about the sum of the multiplicities of the zeros and the degree of the function?

The degree = (sum of multiplicities)

2. Describe the behavior of the graph as it approaches a zero whose multiplicity is 1.

The graph will cross the x-axis.

3. Describe the behavior of the graph as it approaches a zero whose multiplicity is 2.

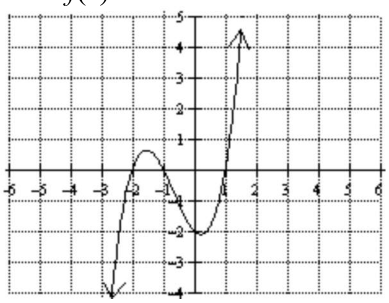
The concavity will NOT change.
The graph will be tangent to x-axis

4. Describe the behavior of the graph as it approaches a zero whose multiplicity is 3.

The graph will cross the x-axis.
The concavity will change.

For each of the following functions, you are given a factor of the function. Identify the zero associated with the given factor. Then, use synthetic division to determine how many times the given factor is a factor of the function. Then, identify the multiplicity of the root associated with that factor.

$f(x) = x^3 + 2x^2 - x - 2$



✓ $\boxed{-1}$ 1 2 -1 -2
0 -1 -1 2

x $\boxed{-1}$ 1 1 -2 | 0
0 -1 0

1 0 | -2

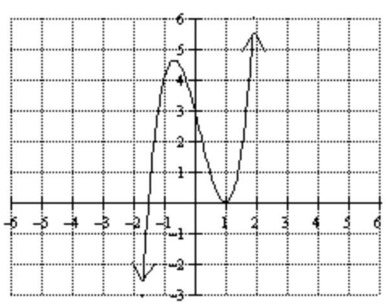
Factor: $(x + 1)$

Root: $x = -1$

Multiplicity: 1 Describe the behavior of the graph at this root.

The graph crosses the x-axis at x = -1 w/o changing concavity

$h(x) = 2x^3 - x^2 - 4x + 3$



✓ $\boxed{1}$ 2 -1 -4 3
0 2 1 -3

✓ $\boxed{1}$ 2 1 -3 | 0
0 2 3

x $\boxed{1}$ 2 3 | 0
0 2

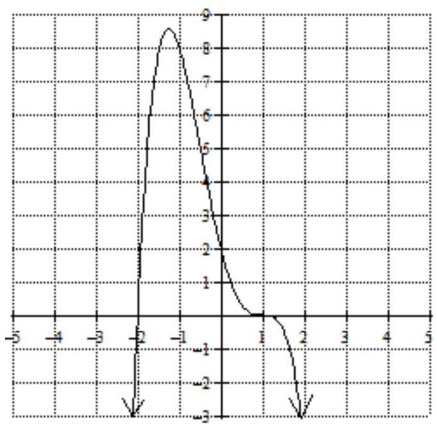
2 | 5

Factor: $(x - 1)$

Root: $x = 1$

Multiplicity: 2 Describe the behavior of the graph at this root.

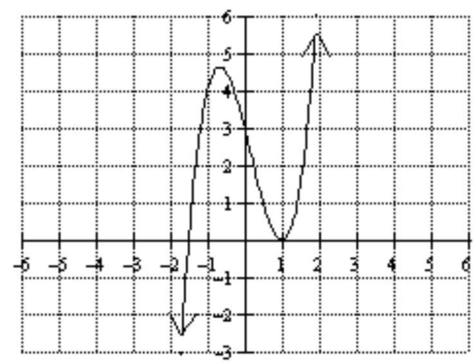
The graph is tangent to the x axis at x = 1

$p(x) = -x^4 + x^3 + 3x^2 - 5x + 2$	
 Factor: $(x-1)$ Root: 1	✓ $\boxed{1}$ -1 1 3 -5 2 / 0 -1 0 3 -2 / -1 0 3 -2 $\lfloor 0$ ✓ $\boxed{1}$ / 0 -1 -1 2 / -1 -1 2 $\lfloor 0$ ✓ $\boxed{1}$ / 0 -1 -2 / -1 -2 $\lfloor 0$ x $\boxed{1}$ / 0 -1 / -1 $\lfloor -3$ Multiplicity: 3 Describe the behavior of the graph at this root. The graph crosses the x-axis at x=1 and changes concavity.

Now, let's consider how we might be able to locate the zeros of a polynomial function numerically.

Consider the function $h(x) = 2x^3 - x^2 - 4x + 3$ that we investigated earlier and whose graph is shown below. Find each pair of function values in the table on your calculator and answer the questions that follow.

Find $h(-2)$ and $h(-1)$.	Find $h(0)$ and $h(2)$.
$h(-2) = -9$ $h(-1) = 4$	$h(0) = 3$ $h(2) = 7$



From the graph, clearly $h(x)$ has a zero between $x = -2$ and $x = -1$. Explain how your finding the values of $h(-2)$ and $h(-1)$ above numerically shows that there is a zero that exists between $x = -2$ and $x = -1$.

- Since $h(-2) = -9 < 0$, the graph of $h(x)$ is below the x-axis at $x=-2$.
- Since $h(-1) = 4 > 0$, the graph of $h(x)$ is above the x-axis at $x=-1$
- $h(x)$ is continuous on $[-2,-1]$ and goes from below the x-axis to above the x-axis

$\therefore$ A zero is guaranteed to exist between $x=-2$ and $x=-1$

Does the same reasoning that you described concerning the zero between $x = -2$ and $x = -1$ hold true for the existence of a zero between $x = 0$ and $x = 2$? Explain your reasoning.

- Since $h(0) = 3 > 0$ and $h(2) = 7 > 0$, the graph of $h(x)$ is above the x-axis at $x=0$ and $x=2$.
- This does not guarantee that a zero exists between $x=0$ and $x=2$.
- (But it is possible for a zero to exist between $x=0$ and $x=2$.)

Based on what we have just seen, what inference can you make about the existence of a zero of a polynomial function if you know the value of the function at two different x – values?

① If f(a) and f(b) are opposite signs, a zero of ODD multiplicity is guaranteed to exist between x=a and x=b

② If f(a) and f(b) are the same signs, a zero of ODD or even multiplicity may exist between x=a and x=b

The table of values below represents values of a quartic function. The function has a negative root and two distinct positive roots. Answer the questions that follow.

x	−4	−1	0	1	2	3	5	6
$F(x)$	40	−8	−3	0	−2	−5	26	90

1. Is/Are any of the roots of $F(x)$ specifically identified in the table? What is the multiplicity of this zero? Explain your reasoning for both questions

F(x) has a zero at x=1. It appears the graph of F(x) is tangent to the x-axis at x=1 because the graph of F(x) is below the x-axis a x=0 and x=2.
The multiplicity of x=1 likely has a multiplicity of 2 because F(x) is quartic with two other sign changes in the table.

2. Between which two x – values in the table is the negative root located? Explain your reasoning.

F(x) has a negative root between x=-4 and x=-1 because F(-4)>0 and F(-1)<0 which means the graph crosses the x-axis between x=-4 and x=-1.

3. Between which two x – values in the table is the second distinct positive root located? Explain your reasoning.

F(x) has a positive root between x=3 and x=5 because F(3)>0 and F(5)<0 which means the graph crosses the x-axis between x=3 and x=5.