Notes 3.2 Solving Polynomial Equations and Inequalities

Algebraic and Graphical Connections

The graph of the function $f(x) = x^3 + x^2 - 6x$ is pictured below. Factor $f(x)$ and use the factors to find the zeros of the function.

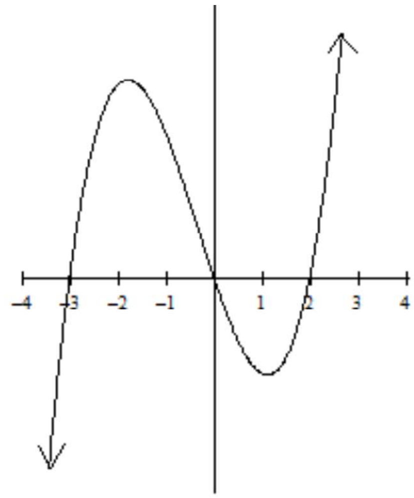
$f(x) = x(x^2 + x - 6)$

$0 = x(x+3)(x-2)$

$x = 0$, $x+3=0 \Rightarrow x=-3$, $x-2=0 \Rightarrow x=2$

Using the graph, identify the value(s) of x for which each of the following equations or inequalities is true.

1. $f(x) = 0$	$x = -3, 0, 2$
2. $f(x) > 0$	$(-3, 0) \cup (2, \infty)$
3. $f(x) < 0$	$(-\infty, -3) \cup (0, 2)$
4. $f(x) \geq 0$	$[-3, 0] \cup [2, \infty)$
5. $f(x) \leq 0$	$(-\infty, -3] \cup [0, 2]$



Let's look at an analytical method by which we could solve a polynomial inequality. It is identical to the method you learned in PreAP Algebra II to solve quadratic inequalities.

1. Completely factor the given polynomial, provided that it is not already factored.	$x^3 + x^2 - 6x \geq 0$ $x(x+3)(x-2) \geq 0$ $x = 0$, $x+3=0 \Rightarrow x=-3$, $x-2=0 \Rightarrow x=2$
2. Set each factor equal to zero to find the zeros of the function. Draw a number line that is divided into segments using the zeros.	$x=-10$ | $x=-1$ | $x=1$ | $x=10$ $(-)(-)(-)$ -3 $(-)(+)(-)$ 0 $(+)(+)(-)$ 2 $(+)(+)(+)$
3. Choose a number from each interval and determine the sign of each factor of the polynomial based on the chosen value. This will enable you to determine if the entire polynomial is positive or negative on the interval in question.	NEG | POS | NEG | POS
4. Refer to the original inequality sign. In this particular example, the inequality symbol is $\geq$ which means the solution will be the values that make the polynomial = 0 or positive. Write the solution interval(s).	$f(x) \geq 0$ means nonnegative $[-3, 0] \cup [2, \infty)$

The graph of the function $g(x) = \underbrace{-x^3 - 3x^2} + \underbrace{2x + 6}$ is pictured below. Factor $g(x)$ and use the factors to determine the zeros of the function.

$g(x) = -x^2(x+3) + 2(x+3)$

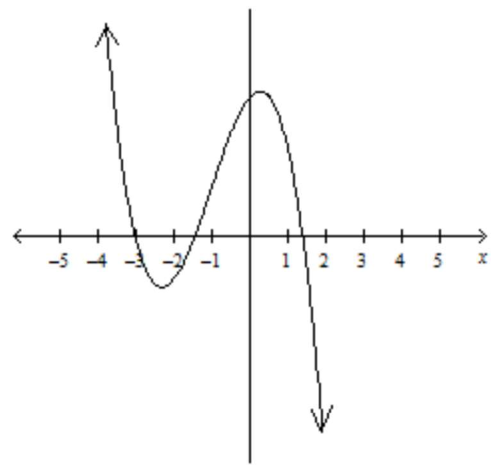
$g(x) = (x+3)(-x^2+2)$

$x+3=0$; $x=-3$

$-x^2+2=0$; $2=x^2$; $\pm\sqrt{2}=x$

Using the graph, identify the value(s) of x for which each of the following equations or inequalities is true.

1. $g(x) = 0$	$x = -3, -\sqrt{2}, \sqrt{3}$
2. $g(x) > 0$	$(-\infty,-3)\cup(-\sqrt{2},\sqrt{2})$
3. $g(x) < 0$	$(-3,-\sqrt{2})\cup(\sqrt{2},\infty)$
4. $g(x) \geq 0$	$(-\infty,-3]\cup[-\sqrt{2},\sqrt{2}]$
5. $g(x) \leq 0$	$[-3,-\sqrt{2}]\cup[\sqrt{2},\infty)$



Now, consider the function $g(x) = -x^3 - 3x^2 + 2x + 6$.

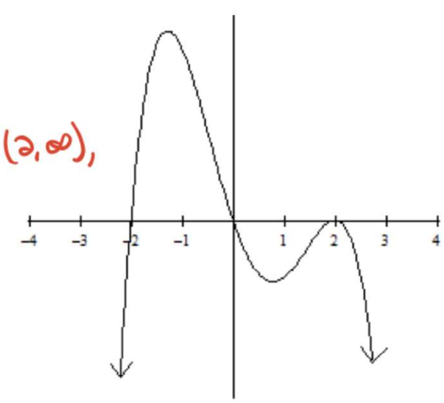
1. Completely factor the given polynomial, provided that it is not already factored.	$-x^3 - 3x^2 + 2x + 6 < 0$ $(x+3)(2-x^2) < 0$ $x = -3, -\sqrt{2}, \sqrt{2}$
2. Set each factor equal to zero to find the zeros of the function. Draw a number line that is divided into segments using the zeros. 3. Choose a number from each interval and determine the sign of each factor of the polynomial based on the chosen value. This will enable you to determine if the entire polynomial is positive or negative on the interval in question.	$x=-10$ $x=-2$ $x=0$ $x=10$ $(-)(-)$ -3 $(+)(-)$ $-\sqrt{2}$ $(+)(+)$ $\sqrt{2}$ $(+)(-)$ POS NEG POS NEG
4. Refer to the original inequality sign. In this particular example, the inequality symbol is $<$ which means the solution will be the values that make the polynomial negative. Write the solution interval(s).	$g(x) < 0$ means negative $(-3,-\sqrt{2})\cup(\sqrt{2},\infty)$

Consider the function $h(x) = -x(x+2)(x-2)^2$.

1. Completely factor the given polynomial, provided that it is not already factored.	$-x(x+2)(x-2)^2 \le 0$ $-x=0$, $x=0$ | $x+2=0$, $x=-2$ | $x-2=0$, $x=2$ | $x-2=0$, $x=2$
2. Set each factor equal to zero to find the zeros of the function. Draw a number line that is divided into segments using the zeros.	$x=-3$ | $x=-1$ | $x=1$ | $x=3$ $(+)(-)(+)$ -2 $(+)(+)(+)$ 0 $(-)(+)(+)$ 2 $(-)(+)(+)$ NEG | POS | NEG | NEG
3. Choose a number from each interval and determine the sign of each factor of the polynomial based on the chosen value. This will enable you to determine if the entire polynomial is positive or negative on the interval in question.	
4. Refer to the original inequality sign. In this particular example, the inequality symbol is $\le$ which means the solution will be the values that make the polynomial = 0 or negative. Write the solution interval(s).	$h(x) \le 0$ means nonpositive $(-\infty,-2] \cup [0,\infty)$

Now, refer back to the graph of $h(x)$ pictured below. Explain how your sign analysis in the process above is confirmed by the graph.

- when $h(x)$ is positive on the interval $(-2,0)$, the graph of $h(x)$ is above the x-axis.
- when $h(x)$ is negative on the interval $(-\infty,-2) \cup (0,2) \cup (2,\infty)$, the graph of $h(x)$ is below the x-axis.



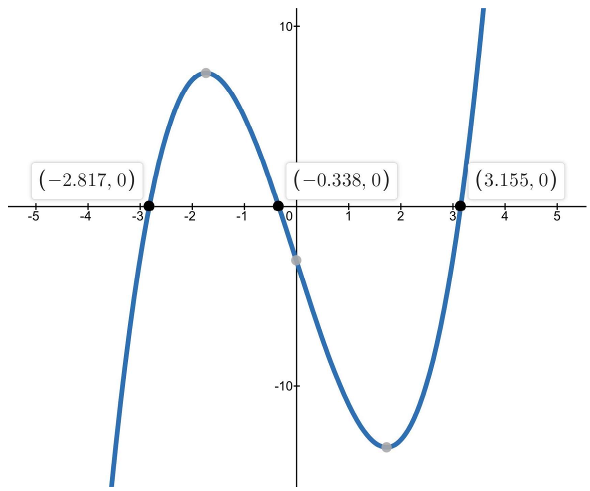
How would your solution to the above inequality be different had the original inequality been $-x(x+2)(x-2)^2 \ge 0$? Explain why and what would the solution be?

If $-x(x+2)(x-2)^2 \ge 0$ means nonnegative which occurs on the interval $[-2,0]$ and $x=2$.

Consider the polynomial inequality $x^3 - 9x \geq 3$. What do you notice about this polynomial as opposed to the previous examples?

- The polynomial is not compared to 0, as it should be.
- Also, $x^3 - 9x - 3 \geq 0$ doesn't appear to be factorable.

At this point in time, not all polynomials are completely factorable by algebraic means. Therefore, in order to solve some polynomial inequalities, we will have to rely on the graphing calculator.

Arrange the inequality so that all of the monomial terms are on one side, compared to 0. Then, graph the function on the calculator. Draw what you see on the grid to the right. $x^3 - 9x - 3 \geq 0$	
Enter $Y_1 = x^3 - 9x - 3$ and use the "zero" function of the calculator to find the three zeros of the function.	$x = -2.817, -0.338, 3.155$
Graphically, what does the solution to the inequality $x^3 - 9x - 3 \geq 0$ represent? What would those intervals of solution be? The solutions to $x^3 - 9x - 3 \geq 0$ represent the x-intercepts.	-2.817 -0.338 3.155 NEG | POS | NEG | POS $[-2.817, -0.338] \cup [3.155, \infty)$